Algebra Candidacy (Fall 2004)

To complete the examination, you will work 6 of the following 12 problems.

1. Show that every group of order 35 is cyclic.

2. Let G be a group with identity e and assume that $g^2 = e$ for every $g \in G$. Show that G is Abelian.

3. Let G be a group such that $G/Z(G)$ is cyclic, where $Z(G)$ denotes the *center* of G. Show that G is Abelian.

4. Let R be a commutative ring. A *derivation* on R is a homomorphism $d\colon\ R \longrightarrow R$ of additive groups satisfying

$$d(a\,b) = a\,d(b) + b\,d(a)$$

for every $a, b \in R$. Show that if d is a derivation on R, then $\mathrm{Ker}(d)$ is a subring of R.

5. Let R be a principal ideal domain. Show that every nonzero prime ideal of R is a maximal ideal.

6. Let R be a ring which is finite. Show that R is an integral domain if and only if R is a field.

7. State the Sylow theorems on subgroups of a finite group.

8. State the fundamental theorem on finitely generated modules over a principal ideal domain.

9. Let $\mathbf{R}$ denote the additive group of real numbers, $\mathbf{Z}$ denote its subgroup of integers, and S denote the multiplicative group of complex numbers of absolute value 1. Show that the quotient group $\mathbf{R}/\mathbf{Z}$ is isomorphic to S.

10. Let S_4 denote the symmetric group on four letters. Describe a normal subgroup N of S_4 and a normal subgroup H of N such that H is NOT a normal subgroup of S_4.

11. (a) Define what is meant by a *gcd* (or greatest common divisor) d of two elements a, b of a principal ideal domain R.

(b) Prove that if R is a principal ideal domain, then any $a, b \in R$ have a gcd.

12. Explicitly describe all the maximal ideals M of the polynomial ring $\mathbf{Z}[X]$ such that $2 + X \in M$, by giving a set of generators for each of them as ideal.